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Solutions in closed form and as power series to the real Lorenz equations

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Abstract
Using the method of the Lie theory of extended groups and for the parameter values $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$ we construct explicitly the general exact solution to the real Lorenz equations in terms of Jacobian elliptic functions. In the context of our approach further possible completely integrable cases of the Lorenz system are discussed by considering the result of the Painlevé analysis for $\sigma = 1$, $b = 2$ and $r = 1/9$ and negative values of r, the latter case, $r < 0$, for $b > 0$ and $\sigma > 0$ not following from the Painlevé test. For other positive parameter values and in the form of appropriate power series we find some particular exact solutions which do not possess the Painlevé property.

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1. Introduction

The system of nonlinear differential equations

$$\dot{x} = \sigma(-x + y) \tag{1.1}$$

$$\dot{y} = rx - y - xz \tag{1.2}$$

$$\dot{z} = -bz + xy \tag{1.3}$$

where the dot denotes differentiation with respect to the time, t, and σ, b and r are non-negative parameters, was proposed by Lorenz [16] in the context of a problem related to meteorology. In recent years the system of differential equations (1.1)–(1.3) has attracted much interest, mainly because of its mathematical properties [3, 6, 21, 22, 25]. In fact systems such as the above are difficult to treat both analytically and numerically. Furthermore they usually depend upon various parameters and exhibit for different ranges of values of the parameters completely different behaviours. For instance in the Lorenz system for $r \gg 1$ and $\sigma \approx 1$ we have a limit cycle whereas for $r \approx \sigma \gg 1$ the solution can be chaotic.

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It is worth noting here that one has to exercise care when adopting numerical results related to nonlinear systems of differential equations [22]. As an example we mention the numerical result that for fixed b and σ the system (1.1)–(1.3) undergoes successive transitions between periodic and aperiodic behaviour as r increases [17], a result not confirmed for $b = 2\sigma$, $b =$ constant and increasing $r > 1$ in the framework of an analysis of (1.1)–(1.3) by means of the theory of nonlinear differential equations [3].

With the exception of studies dealing with the Painlevé property [25] and the construction of first integrals [21] of the Lorenz system most of the work done on the system (1.1)–(1.3) is mainly numerical. However, it is a rather common knowledge that sometimes, as noted above, numerical investigations of coupled nonlinear differential equations may fail to reveal important aspects of the behaviour of their solutions. Motivated by the above fact and since analytic solutions do provide a reliable test for the accuracy and feasibility of numerical algorithms, we initiate in the present paper an analytical study of the Lorenz equations (1.1)–(1.3).

Provided the parameters b and σ fulfil the constraint $b = 2\sigma$, the Lorenz system is reduced essentially either: (i) to a generalized Emden–Fowler equation or; (ii) to a time-dependent oscillator with constant coefficient anharmonic term. By utilizing the vehicle of the Lie theory of extended groups [5, 9] in approach (i) or the Painlevé analysis [19] in approach (ii) we explicitly construct the general exact solution to equations (1.1)–(1.3) valid for $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$. Further the possible existence of completely integrable cases of the system (1.1)–(1.3) for the parameter sets ($\sigma = 1, b = 2, r = 1/9$) and ($b = 2\sigma, \sigma > 0, r < 0$), the $r < 0$ case not being covered by a Painlevé analysis of equations (1.1)–(1.3), are discussed in the context of approach (i). The extension of the range of applicability of results originating from a Painlevé analysis in approach (ii) allows us to find for other positive parameter values and expressed as power series some particular exact solutions not possessing the Painlevé property. The connection to previous analytic results is discussed wherever it is appropriate.

The results which we obtain indicate both the power and the limitations of the Painlevé analysis. When it works (the precise conditions can be found, for example, in Conte [1]), we know that the system under consideration has a solution in terms of functions analytic except at pole-like singularities (algebraic branch points in the case of the so-called 'weak' Painlevé property). A standard criterion for the possession of the Painlevé property is that all possible patterns of singular behaviour pass the Painlevé test [24] and yet there exist instances [11] for which this criterion is not satisfied and yet the solution of the system is manifestly analytical except for its correct singularities. Some of the results presented here, as has been reported in divers contexts [8, 10, 12, 15], emphasize the point that a satisfactory definition of integrability persists in remaining elusive.

2. Reduction of the Lorenz system to an Emden–Fowler equation

From the Lorenz equations (1.1)–(1.3) we deduce the equivalent system

$$\ddot{x} + (\sigma + 1)\dot{x} + \sigma(1 - r)x = -\sigma x z \tag{2.1}$$

$$\dot{z} + bz = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x^2}{2\sigma}\right) + x^2 \tag{2.2}$$

$$\dot{x} = \sigma(-x + y). \tag{2.3}$$

Equation (2.2) yields

$$z(t) = C\exp(-bt) + \frac{x^2}{2\sigma} + \exp(-bt)\left(1 - \frac{b}{2\sigma}\right)\int x^2 \exp(bt)\,\mathrm{d}t \tag{2.4}$$

where C is a constant of integration. Equation (2.4) is inserted into equation (2.1) to give

$$\ddot{x} + (\sigma + 1)\dot{x} + \sigma(1 - r)x = -\exp(-bt)C\sigma x - \tfrac{1}{2}x^3 - x[\exp(-bt)](\sigma - \tfrac{1}{2}b)\int x^2 \exp(bt)\,\mathrm{d}t. \qquad (2.5)$$

At this stage we do not attempt to treat the integrodifferential equation (2.5). Instead in the following we assume that

$$b = 2\sigma. \qquad (2.6)$$

The relation (2.6) between the parameters b and σ also appears in the construction of first integrals to equations (1.1)–(1.3) [21]. In this paper it is of importance since all the solutions we are going to construct hold subject to (2.6). Due now to equations (2.6), (2.5) becomes

$$\ddot{x} + (\sigma + 1)x + \sigma(1 - r)x = -\exp(-bt)C\sigma x - \tfrac{1}{2}x^3. \qquad (2.7)$$

We introduce the functions $u(\phi)$, $\phi = \phi(t)$ and $v(t)$ through

$$x(t) = u(\phi)v(t) \qquad \phi = \phi(t). \qquad (2.8)$$

The transformation (2.8) is inserted into equation (2.7) to give

$$v(\dot{\phi})^2u'' + u'[v\ddot{\phi} + 2\dot{v}\dot{\phi} + (\sigma + 1)v\dot{\phi}] + u[\ddot{v} + (\sigma + 1)\dot{v} + \sigma(1 - r)v + \exp(-bt)C\sigma v] = -\tfrac{1}{2}u^3v^3 \qquad (2.9)$$

where the prime denotes differentiation with respect to ϕ. We require now that

$$v\ddot{\phi} + 2\dot{v}\dot{\phi} + (\sigma + 1)v\dot{\phi} = 0 \qquad (2.10)$$

$$\ddot{v} + (\sigma + 1)\dot{v} + \sigma(1 - r)v + \exp(-bt)C\sigma v = 0. \qquad (2.11)$$

Equation (2.11) is solved by (particular solution)

$$v(t) = \exp\left[-\frac{1}{2}(\sigma + 1)t\right] Z_p\left[2\frac{\sqrt{C\sigma}}{b}\exp\left(-\frac{1}{2}bt\right)\right] \qquad (2.12)$$

where

$$p = \frac{1}{b}[(\sigma - 1)^2 + 4\sigma r]^{1/2} \qquad b = 2\sigma \qquad (2.13)$$

and Z_p is a cylinder function of order p. Equation (2.10) yields (particular solution)

$$\dot{\phi} = \exp[-(\sigma + 1)t]/v^2. \qquad (2.14)$$

Due to equations (2.10)–(2.14) equation (2.9) becomes

$$u''(\phi) = -\tfrac{1}{2}u^3\exp[-(\sigma + 1)t]Z_p^6(\exp(-\sigma t)(C/\sigma)^{1/2}). \qquad (2.15)$$

In this section we consider the case $p = 1/2$. Then equation (2.13) gives

$$r = (2\sigma - 1)/4\sigma \qquad b = 2\sigma \qquad (2.16)$$

and

$$Z_{1/2}(\zeta) = J_{1/2}(\zeta) = (2/\pi\zeta)^{1/2}\sin\zeta \qquad \zeta = (C/\sigma)^{1/2}\exp(-\sigma t) \qquad (2.17)$$

where $J_{1/2}(\zeta)$ is the Bessel function of the first type. Equation (2.14) becomes, due to equations (2.12) and (2.17) after integration and up to an inessential constant,

$$\phi(t) = (\pi/2\sigma)\cot\zeta \qquad \zeta = (C/\sigma)^{1/2}\exp(-\sigma t) \quad C > 0 \qquad (2.18)$$

the constraint $C > 0$ following from the requirement for real solutions. Equation (2.15) now gives

$$u''(\phi) = -\lambda u^3\exp[(2\sigma - 1)t]\sin^6\zeta \qquad \lambda = (4/\pi^3)(\sigma/C)^{3/2}. \qquad (2.19)$$

To simplify equation (2.19) further we set $2\sigma - 1 = 0$. Thus equation (2.16) implies

$$\sigma = \tfrac{1}{2} \qquad b = 1 \qquad \text{and} \qquad r = 0. \tag{2.20}$$

Finally equation (2.19) becomes by virtue of equations (2.18) and (2.20)

$$F''(p) = -\epsilon F^3(p)/(1+p^2)^3 \qquad \epsilon = (l/2C)^{3/2}(4/\pi) \tag{2.21}$$

where

$$F(p) = u(\phi) \qquad \phi = \pi p \qquad p = \cot[(2C)^{1/2}\exp(-t/2)]. \tag{2.22}$$

Equation (2.21) has the form of a generalized Emden–Fowler equation [13,14]. In the following section we show that equation (2.21) is amenable to a treatment in the context of the Lie theory of extended groups [5, 9]. Thus we are able to construct the general exact solution to equations (1.1)–(1.3) as announced in the introduction.

3. General exact solution for $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$

Equation (2.21) has a Lie point symmetry

$$G = \xi(p, F)\partial_p + \eta(p, F)\partial_F \qquad F = F(p) \tag{3.1}$$

if [9]

$$G^{[2]}N(F'', F, p)_{|N=0} = 0 \tag{3.2}$$

where $N = 0$ is (2.21), the second extension of G, denoted by $G^{[2]}$, is given by

$$G^{[2]} = G + (\eta' - \xi F')\partial_{F'} + (\eta'' - \xi'' F' - 2\xi' F'')\partial_{F''} \tag{3.3}$$

and the prime denotes differentiation with respect to p.

Since both ξ and η are functions of p and F only, we obtain from equation (3.2) after separating coefficients of F'^3, F'^2 and F' the determining equations

$$\frac{\partial^2\xi}{\partial F^2} = 0 \tag{3.4}$$

$$\frac{\partial^2\eta}{\partial F^2} - 2\frac{\partial^2\xi}{\partial F\partial p} = 0 \tag{3.5}$$

$$2\frac{\partial^2\eta}{\partial F\partial p} - \frac{\partial^2\xi}{\partial p^2} + 3g\frac{\partial\xi}{\partial F} = 0 \tag{3.6}$$

$$\frac{\partial^2\eta}{\partial p^2} - g\frac{\partial\eta}{\partial F} + 2g\frac{\partial\xi}{\partial p} + \xi\frac{\partial g}{\partial p} + \eta\frac{\partial g}{\partial F} = 0 \tag{3.7}$$

the solution of which yields the single symmetry

$$G = (p^2 + 1)\partial_p + pF\partial_F. \tag{3.8}$$

Now to transform (2.21) to a p-free equation we seek the transformation $(p, F) \longrightarrow (P, f)$:

$$P = \phi_1(p, F) \qquad f = \phi_2(p, F), \tag{3.9}$$

which converts (3.8) to ∂_P. This is achieved if

$$(1 + p^2)\frac{\partial\phi_1}{\partial p} + pF\frac{\partial\phi_1}{\partial F} = 1 \tag{3.10}$$

$$(1 + p^2)\frac{\partial\phi_2}{\partial p} + pF\frac{\partial\phi_2}{\partial F} = 0. \tag{3.11}$$

From the characteristics of (3.10) and (3.11) we select the solutions

$$P = \phi_1(p, F) = \arctan p \qquad f = \phi_2(p, F) = F(1 + p^2)^{-1/2}. \tag{3.12}$$

Under the transformation (3.12) equation (2.21) becomes

$$\frac{\mathrm{d}^2 f}{\mathrm{d}P^2} = -\epsilon f^3(P) - f(P). \tag{3.13}$$

Equation (3.13) has a first integral

$$\left(\frac{\mathrm{d}f}{\mathrm{d}P}\right)^2 + f^2 + \frac{1}{2}\epsilon f^4 = C_1 \qquad C_1 > 0. \tag{3.14}$$

(We are informed that the same first integral is reported by Polyanin and Zaitsev [18, equation (9), p 421], being derived by a method which is purported not to be based on symmetry.) Integration of (3.14) yields

$$P - P_0 = \left(\frac{2}{\epsilon}\right)^{1/2} \int_{f_0}^{f} \left[\left(A^2 - f_1^2\right)\left(B^2 + f_1^2\right)\right]^{-1/2} \mathrm{d}f_1 \tag{3.15}$$

where

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}\left[\left(\left(\frac{2}{\epsilon}\right)^2 + 8\frac{C_1}{\epsilon}\right)^{1/2} - \frac{2}{\epsilon}\right]^{1/2} \\ B &= \frac{1}{\sqrt{2}}\left[\left(\left(\frac{2}{\epsilon}\right)^2 + 8\frac{C_1}{\epsilon}\right)^{1/2} + \frac{2}{\epsilon}\right]^{1/2} \\ f(P_0) &= f_0 \end{aligned} \tag{3.16}$$

and

$$f^2 \leqslant A^2. \tag{3.17}$$

Now (3.15) becomes [7]

$$P - P_0 = (A^2 + B^2)^{-1/2}\left(\frac{2}{\epsilon}\right)^{1/2} [F(\gamma_1, \delta) - F(\gamma_2, \delta)] \tag{3.18}$$

where

$$\begin{aligned} \gamma_1 &= [\arcsin(f/A)]\left[\frac{(A^2 + B^2)}{(B^2 + f^2)}\right]^{1/2} \\ \gamma_2 &= [\arcsin(f_0/A)]\left[\frac{(A^2 + B^2)}{(B^2 + f_0^2)}\right]^{1/2} \\ \delta &= \frac{A}{(A^2 + B^2)^{1/2}} \end{aligned} \tag{3.19}$$

and $F(\gamma_1, \delta)$ and $F(\gamma_2, \delta)$ are elliptic integrals of the first kind. We note that equation (3.18) holds for $0 < f_0 \leqslant A$, $0 < f \leqslant A$. If, for example, $f_0 < 0$, we may use the relation $F(\gamma_2, \delta) = -F(-\gamma_2, \delta)$. Thus in the following we assume without loss of generality that

$$0 < f_0 < f \tag{3.20}$$

the f being also subject to the constraint (3.17). To invert equation (3.18) we use the standard relation

$$\sin\gamma_1 = \mathrm{sn}\Lambda \tag{3.21}$$

where

$$\Lambda = \Lambda(P) = (P - P_0)[(A^2 + B^2)\epsilon/2]^{1/2} + F(\gamma_2, \delta) \tag{3.22}$$

and snΛ is the Jacobian elliptic function. From equations (3.19) and (3.21) we deduce

$$f(P) = (AB\mathrm{sn}\Lambda)(B^2 + A^2\mathrm{cn}^2\Lambda)^{-1/2} \tag{3.23}$$

where automatically sn$\Lambda > 0$, due to equations (3.19)–(3.21) and the constraint (3.17) is fulfilled.

Having now solved (3.13) we are in a position to construct the general exact solution to equation (2.7) for $\sigma = \frac{1}{2}, b = 1, r = 0$. Recalling first that in equation (2.8) we need $v(t)$ and $\phi(t)$, we find $v(t)$ from equations (2.12), (2.16), (2.17) and (2.20)

$$v(t) = \exp(-t/2)\sin[(2C)^{1/2}\exp(-t/2)](2/\pi^2 C)^{1/4} \tag{3.24}$$

and $\phi(t)$ from equation is (2.18) and (2.20)

$$\phi(t) = \cot[(2C)^{1/2}\exp(-t/2)]\pi. \tag{3.25}$$

Further from equations (2.22), (3.12) and (3.23) we obtain

$$u(\phi) = \frac{AB\mathrm{sn}[(\arctan\phi/\pi - P_0)((A^2 + B^2)\epsilon/2)^{1/2} + F(\gamma_2, \delta)](1 + \phi^2/\pi^2)^{1/2}}{\left\{B^2 + A^2\mathrm{cn}^2[(\arctan\phi/\pi - P_0)((A^2 + B^2)\epsilon/2)^{1/2} + F(\gamma_2, \mathrm{d})]\right\}^{1/2}}. \tag{3.26}$$

Finally (2.8) and (3.24)–(3.26) yield the general exact solution to equation (2.7). Since the expression for $x(t)$ obviously contains three constants of integration and $y(t)$ and $z(t)$ are immediately obtainable from equations (2.3), (2.4) and (2.6), we have the general exact solution of the Lorenz system (1.1)–(1.3) for $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$ (equation(2.20)) as announced in the introduction to this paper. It is also clear that $x(t)$, $y(t)$ and $z(t)$ tend asymptotically to zero for $t \to \infty$, that is the solution found $(x(t), y(t), z(t))$ of the Lorenz equations approaches the stable (since $r = 0$) equilibrium point $x = y = z = 0$. One may notice that $x(t)$ has the form of a damped oscillation due to the appearance of snΛ, a result to be expected since the basic equation, (2.7), is a Duffing's equation without the driving term.

The important point that should be stressed here is that by constructing the general exact solution of equations (1.1)–(1.3) for $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$ we have actually proved directly that the system of differential equations (1.1)–(1.3) is algebraically completely integrable for the above parameter values. Since the Lorenz system passes the Painlevé test [23] for $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$ [20], our result conforms with the existing theorems [23] linking the necessity of passing the Painlevé test with algebraic complete integrability through Abelian functions. This is indeed so since the general exact solution found possesses the Painlevé property as it is given in terms of Jacobian elliptic functions, the singularities of which in the complex time plane are only simple poles [23].

We recall here that by means of a rescaling [23, 25] it has been shown that the Lorenz system is algebraically completely integrable by means of elliptic functions for $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$ and $t \to \infty$ with $x(t) \to 0$, $y(t) \to 0$ and $z(t) \to 0$. Evidently this is just the asymptotic case which is included in our general exact solution as we mentioned above.

Considering now the basic equation (2.7) and setting $C = 0$ we can construct [4], by solving the resulting differential equations, a particular exact solution of the Lorenz system again by means of Jacobian elliptic functions. This particular solution depends clearly on two constants of integration and is valid for $b = 2\sigma$ and $\frac{1}{2} \leqslant \sigma \leqslant 2$, $0 \leqslant r \leqslant 1/9$. At this point it is probably worth noting that the price we have to pay to show complete integrability of the Lorenz equations is the restriction to one point in the parameter space, i.e. $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$, while the particular exact solution found earlier [4] holds for a wider range of parameters. We return to this observation in section 4 of this paper.

In the following we indicate how the particular exact solution [4] of equations (1.1)–(1.3) can be obtained in the context of this paper.

On setting $C = 0$ in equation (2.7) and, consequently, in (2.11) we obtain for $v(t)$

$$v(t) = \exp(\mu t) \tag{3.27}$$

where μ is a solution of the equation $\mu^2 + (\sigma + 1)\mu + \sigma(1 - r) = 0$, which for $\sigma > 0$ and $r \geqslant -(\sigma - 1)^2/4\sigma$ always has real roots. Thus (2.14) yields

$$\dot{\phi} = \exp[-t(\sigma + 1 + 2\mu)]. \tag{3.28}$$

By virtue of equations (3.27) and (3.28) equation (2.9) becomes

$$u'' \exp[t(-2\sigma - 2 - 3\mu)] = -\tfrac{1}{2}u^3 \exp(3\mu t). \tag{3.29}$$

Equation (3.29) is integrable by means of elliptic functions provided $-2\sigma - 2 - 3\mu = 3\mu$, which, since $\mu^2 + (\sigma + 1)\mu + \sigma(1 - r) = 0$ and remembering that $\sigma > 0$, is written as

$$2(\sigma + 1)^2 = 9\sigma(1 - r). \tag{3.30}$$

Equation (3.30) gives, if we restrict ourselves to $r \geqslant 0$, as mentioned previously, the relations $\frac{1}{2} \leqslant \sigma \leqslant 2$, $0 \leqslant r \leqslant 1/9$ with $b = 2\sigma$. The function $x(t)$ is given, due to equations (2.8) and (3.27)–(3.29), by

$$x(t) = u(\phi) \exp[-\tfrac{1}{3}(\sigma + 1)t] \qquad \phi(t) = -3\frac{\exp[-\frac{1}{3}(\sigma + 1)t]}{\sigma + 1}. \tag{3.31}$$

The particular exact solution to the Lorenz system can now easily be written down. We note that, as one would expect, condition (3.30) also follows from the requirement that equation (2.7) with $C = 0$ passes the Painlevé test [23], a result which is obvious in the framework of our method from equations (2.8) and (3.27)–(3.30). Furthermore the above particular solution ($x(t)$, $y(t)$ and $z(t)$) approaches the stable (since $r = 0$) equilibrium point $x = y = z = 0$.

Finally we point out that only for $\sigma = \frac{1}{2}$, $b = 1$ and $r = 0$ can the aforementioned particular solution with two constants of integration be obtained from the general exact solution. To do this we let $C \to 0$ in equation (2.21) and we obtain by virtue of (2.22)

$$F''(p) + \epsilon\frac{F^3(p)}{p^6} = 0 \qquad p \to \infty \quad \epsilon \to \infty. \tag{3.32}$$

Equation (3.32) is transformed by means of the substitution

$$F(p) = pW(1/p) \tag{3.33}$$

into

$$W''(R) + \epsilon W^3(R) = 0 \quad R = \frac{1}{p} \quad R \to 0 \qquad \epsilon \to \infty. \tag{3.34}$$

From the definition of p in equations (2.22) and (3.33) we get

$$\begin{aligned} &F(p) = \exp(t/2)(2C)^{-1/2}W(R) \\ &p = \exp(t/2)(2C)^{-1/2} \qquad R = \frac{l}{p} \quad C \to 0 \quad R \to 0. \end{aligned} \tag{3.35}$$

Thus equations (2.8), (2.22), (3.24) and (3.35) yield for $x(t)$

$$x(t) = \exp(-t/2)(2/C)^{1/4}\pi^{-1/2}W(R) \qquad R = \frac{1}{p} \quad C \to 0 \quad R \to 0. \tag{3.36}$$

Now we introduce a function

$$m(\tau) = kW(R) \qquad \tau = -2\exp(-t/2) \qquad R = -\tau(C/2)^{1/2} \qquad R \to 0. \tag{3.37}$$

Owing to the definition of ϵ in (2.21) and to (3.34) we choose $k = (2/\pi)^{1/2}(2C)^{-1/4}$ and so

$$m''(\tau) + \tfrac{1}{2}m^3(\tau) = 0 \tag{3.38}$$

while $x(t)$ in equation (3.36) becomes

$$x(t) = \exp(-t/2)m(\tau) \qquad \tau = -2\exp(-t/2). \tag{3.39}$$

On setting $\sigma = 1$, $b = 1$ and $r = 0$ in equations (3.29)–(3.31) we conclude that the $x(t)$ defined by equation (3.31) is identical with that defined by equations (3.38) and (3.39).

4. Investigation of other cases

Suppose we chose $p = \frac{3}{2}$ in equation (2.15). Then equation (2.13) gives

$$r = (4\sigma - 1)(2\sigma + 1)/4\sigma \qquad b = 2\sigma \tag{4.1}$$

and

$$\begin{aligned} Z_{3/2}(\zeta) &= J_{3/2}(\zeta) = 2^{1/2}(\pi\zeta)^{-1/2}\left(\frac{\sin\zeta}{\zeta} - \cos\zeta\right) \\ \zeta &= \left(\frac{C}{\sigma}\right)^{1/2}\exp(-\sigma t) \end{aligned} \tag{4.2}$$

where $J_{3/2}$ is the Bessel function of the first kind. Equation (2.14) yields, owing to equations (2.12) and (4.2), after integration

$$\phi(t) = -\frac{\pi(\zeta\sin\zeta + \cos\zeta)}{2\sigma(\zeta\cos\zeta - \sin\zeta)} \tag{4.3}$$

by application of the formula [7]

$$\int \frac{\zeta^2\,d\zeta}{(\zeta\cos\zeta - \sin\zeta)^2} = \frac{\zeta\sin\zeta + \cos\zeta}{\zeta\cos\zeta - \sin\zeta}. \tag{4.4}$$

Consequently equation (2.15) becomes, after some algebra,

$$\begin{aligned} u''(\phi) &= -\frac{\epsilon_1 u^3(\phi)\exp[t(8\sigma - 1)](\zeta^2 + 1)^3}{\pi^6(4\sigma^2\phi^2/\pi^2 + 1)^3} \\ \epsilon_1 &= 4\pi^3(\sigma/C)^{9/2}. \end{aligned} \tag{4.5}$$

We simplify (4.5) by setting $8\sigma - 1 = 0$, i.e. $\sigma = 1/8$. Thus equation (4.1) shows that $r = -5/4$, that is we obtain $r < 0$. At this point we have two alternatives, either to give up further treatment of equation (4.5) since, in the original context in which the Lorenz equations were derived, we have $r > 0$, or to consider equations (1.1)–(1.3) as a dynamical system in the properties of which we are interested and, thus, to also allow $r < 0$. We choose the second alternative and equation (4.5) gives

$$u''(\phi) = -\frac{\epsilon_1 u^3(\phi)(\zeta^2 + 1)^3}{\pi^6(\phi^2/16\pi^2 + 1)^3} \qquad \epsilon_1 = 4\pi^3(1/8C)^{9/2}. \tag{4.6}$$

We have not been able to invert equation (4.3) to eliminate ζ from equation (4.6). Nevertheless we are able to solve equation (4.6) in the limit as $t \to \infty$, which by virtue of equations (4.2) and (4.3) corresponds to $\zeta \to 0$, $\phi \to \infty$. Hence (4.6) is written as

$$u''(\phi) = -\epsilon_1(16)^3 u^3(\phi)/\phi^6 \qquad \phi \to \infty. \tag{4.7}$$

On introducing into (4.7) the function $W_1(1/\phi)$ through

$$u(\phi) = \phi W_1(1/\phi) \tag{4.8}$$

we find that (4.7) becomes

$$W_1''(\Phi) + (16)^3\epsilon_1 W_1^3(\Phi) = 0 \qquad \Phi = 1/\phi \tag{4.9}$$

which is immediately integrable in terms of elliptic functions.

The case $p = 5/2$ in equation (2.15) yields

$$r = (24\sigma^2 + 2\sigma - 1)/4\sigma \qquad b = 2\sigma \tag{4.10}$$

$$\phi(t) = k_1 \frac{3\cos\zeta + 3\zeta\sin\zeta - \zeta^2\cos\zeta}{3\sin\zeta - 3\zeta\cos\zeta - \zeta^2\sin\zeta} \tag{4.11}$$

$$u'' = -\frac{\epsilon_2 u^3(\phi)\exp[t(14\sigma - 1)](9 + 3\zeta^2 + \zeta^4)^3}{(k_2\phi^2 + 1)^3} \tag{4.12}$$

where k_1, ϵ_2 and k_2 are positive constants. Proceeding along the lines of equation (4.5) we set in equation (4.12) $14\sigma - 1 = 0$, i.e. $\sigma = 1/14$, thus obtaining from (4.10) $r = -18/7$. The resulting equation from (4.12), due to the difficulty of inverting equation (4.11) can then be solved in the limit as $t \to \infty$, i.e. $\zeta \to 0$ and $\phi \to -\infty$ by means of elliptic functions.

We now recall that the limit $\zeta \to 0$ can also be achieved by letting $C \to 0$ in (4.2), but $C \to 0$ implies that we actually solve equation (2.7) with $C = 0$ and, thus, find particular solutions to the Lorenz system, as we saw in section 3. These particular solutions to equations (1.1)–(1.3) correspond to the following two cases here:

$$\begin{array}{lllll} \text{Case} \quad p = 3/2 & \text{i.e.} \quad \sigma = 1/8 & b = 1/4 & r = -5/4 \\ \text{Case} \quad p = 5/2 & \text{i.e.} \quad \sigma = 1/14 & b = 1/7 & r = -18/7. \end{array} \tag{4.13}$$

By solving equations (4.7)–(4.9) and the corresponding equations, which result from (4.12) for $\sigma = 1/14$, $\zeta \to 0$, while in both cases we have $C \to 0$, we can obtain the above particular solutions in terms of elliptic functions and for the parameter values (4.13). The procedure we follow is identical to the one we used in extracting the particular solution to equations (1.1)–(1.3) for $\sigma = 1/2$, $b = 1$, $r = 0$ from the relevant general solution (equations (3.32)–(3.39)).

At this point we stress two important aspects of the above procedure. Firstly, we utilize in all three p-cases the basic differential equations (2.21) for $p = 1/2$, (4.6) for $p = 3/2$ and (4.12) for $p = 5/2$, the general solutions of which give us essentially the general exact solution to the Lorenz system for the relevant parameter values (2.20) and (4.13) respectively. In other words we do not need the analytic form of the general solution, which for the time being is known only for $\sigma = 1/2$, $b = 1$, $r = 0$, to construct the respective particular exact solution. Secondly, all three particular solutions are expressed by means of elliptic functions and, consequently, they are also obtainable in the framework of equations (3.27)–(3.30). In fact the parameter values (4.13) satisfy condition (3.30).

Thus in summary we have the following. For (σ, b, r) fulfilling relation (3.30) equation (2.7) passes for $C = 0$ the Painlevé test, as noted previously, and is integrable through elliptic functions. However, out of all possible (σ, b, r) values given by (3.30), only for $(\sigma = l/2, b = 1, r = 0)$, $(\sigma = 1/8, b = 1/4, r = -5/4)$ and $(\sigma = 1/14, b = 1/7, r = -18/7)$ have we been able to find differential equations the general solutions of which lead us immediately to general exact solutions of the Lorenz system and, furthermore, only for $\sigma = l/2$, $b = 1$, $r = 0$ could we solve by means of Jacobian elliptic functions the relevant differential equation (2.21).

We now observe that for (σ, b, r) given by equation (4.13) the Lorenz equations do not pass the Painlevé test. Nevertheless there are dynamical systems [23] which do not pass the Painlevé test and yet are algebraically completely integrable. Thus, in view of the common features exhibited by the particular and general solutions to the Lorenz system for the parameter values (2.20), (4.13), as noted above, and in spite of the apparent difficulty in inverting equations (4.3) and (4.11), we conjecture that equations (4.3) and (4.6) on the one hand and equations (4.11) and (4.12) ($\sigma = 1/14$) on the other are integrable through functions which are generalizations of Jacobian elliptic functions. These generalizations, however complicated they may be, should degenerate for $C \to 0$ to Jacobian elliptic functions. Our hypothesis includes possibly further (σ, b, r) values, which are generated by considering higher p-values in equation (2.15) of the form $p = (2n+1)/2$ with $n = 0, 1, 2, 3, \ldots$.

In conclusion we make three further points. Firstly, the consideration of the Bessel function of the second kind $N_p(\zeta)$ in equation (2.15) as well as the introduction of linear combinations of cylinder functions in equation (2.12) and following do not yield new results, as probably one would expect. Secondly, apart from the case ($\sigma = 1/2, b = 1, r = 0$), equations (1.1)–(1.3) pass the Painlevé test for ($\sigma = 1/3, b = 0, r =$ arbitrary) and ($\sigma = 1, b = 2, r = 1/9$). The case ($\sigma = 1/3, b = 0, r =$ arbitrary) is not included in our approach due to the constraint we impose on b and σ, i.e. $b = 2\sigma$. However, the case ($\sigma = 1, b = 2, r = 1/9$) implies by virtue of equation (2.13) that $p = 1/3$, which leads us to using Airy functions in the respective calculations. Thirdly, one could envisage the construction of closed form solutions to equations (1.1)–(1.3) for all $b > 0, \sigma > 0$ with $b = 2\sigma$. This could be attempted by considering equations (2.12)–(2.15) in which case we should first calculate

$$\begin{aligned}\phi(t) &= -\frac{1}{\sigma}\int \frac{d\zeta}{\zeta J_p^2(\zeta)} \qquad (\zeta = (C/\sigma)^{1/2}\exp(-\sigma t)) \\ &= \frac{\pi}{2\sigma\sin(p\pi)}\frac{J_{-p}(\zeta)}{J_p(\zeta)}\end{aligned} \tag{4.14}$$

where $J_p(\zeta)$ is the Bessel function of the first kind, invert equation (4.14), insert the result into equation (2.15) and solve the resulting generalized Emden–Fowler equation. Although the computational difficulties encountered in such a program may be hard to surmount, at least we have an analytical procedure which in principle can be followed.

5. Non-Painlevé particular solutions

Our starting point is equation (2.9) which we convert to a time-dependent oscillator with constant coefficient anharmonic force [2]. Thus we require firstly that (2.10) and, hence, (2.14) hold. Then for the coefficient of u^3 in the differential equation we seek to make a constant k and set

$$-k = -\frac{1}{2}\frac{v^2}{\dot{\phi}^2} \qquad k > 0 \tag{5.1}$$

since we require $\phi(t)$ and $v(t)$ to be real. Equations (2.14) and (5.1) yield

$$v(t) = (2k)^{1/6}\exp[-\tfrac{1}{3}(\sigma+1)t] \tag{5.2}$$

so that (2.14) gives

$$\phi(t) = -\frac{3\exp[-\frac{1}{3}(\sigma+1)t]}{(2k)^{1/3}(\sigma+1)}. \tag{5.3}$$

The coefficient of u now becomes, due to equations (2.9), (5.2) and (5.3),

$$\frac{1}{v(\dot{\phi})^2}[\ddot{v} + (\sigma+1)\dot{v} + \sigma(1-r)v + C\sigma v\exp(-2\sigma t)] = \frac{A}{\phi^2} + B\phi^{(4\sigma-2)/(\sigma+1)} \tag{5.4}$$

where

$$A = \frac{[9\sigma(1-r) - 2(\sigma+1)^2]}{(\sigma+1)^2} \tag{5.5}$$

$$B = \frac{9C\sigma[-(2k)^{1/3}(\sigma+1)/3]^{6\sigma/(\sigma+1)}}{(\sigma+1)^2}. \tag{5.6}$$

Due to equations (2.10), (5.1) and (5.4) equation (2.9) is written as

$$u''(\phi) + \left(\frac{A}{\phi^2} + B\phi^{(4\sigma-2)/(\sigma+1)}\right)u(\phi) + u^3(\phi) = 0. \tag{5.7}$$

For a Painlevé analysis of equation (5.7) we let

$$u = \alpha\phi^p + \mu\phi^{p+k}. \tag{5.8}$$

On substituting (5.6) into (5.7) and keeping terms to first order in b we obtain

$$\alpha p(p-1)\phi^{p-2} + \mu(p+k)(p+k-1)\phi^{p+k-2} + A\alpha\phi^{p-2} + A\mu\phi^{p+k-2}$$
$$+B\alpha\phi^{p+(4\sigma-2)/(\sigma+1)} + B\mu\phi^{p+k+(4\sigma-2)/(\sigma+1)} + \alpha^3\phi^{3p} + 3\alpha^2\mu\phi^{3p+k} = 0. \tag{5.9}$$

On ignoring μ we obtain that the powers of the leading terms balance when either

$$3p = p - 2 \tag{5.10}$$

or

$$p + (4\sigma - 2)/(\sigma + 1) = 3p. \tag{5.11}$$

For possible balance of all leading terms we get either from equation (5.10) or equation (5.11) that $\sigma = 0$ which is trivial and can be ignored. Furthermore, for equation (5.11) to correspond to the dominant terms we must also have, due to (5.10), $3p < p - 2$ which leads to $-1 < \sigma < 0$. In this paper we consider only $\sigma > 0$ (see section 3) and, thus, we are left with equation (5.10) which gives $p = -1$. Therefore the coefficients of the leading terms balance if

$$2\alpha + A\alpha + \alpha^3 = 0. \tag{5.12}$$

Equation (5.12) shows that

$$\alpha^2 = -(2 + A). \tag{5.13}$$

The resonances follow from equation (5.9) by considering the first order terms in μ. We find, therefore, that for equation (5.7) the Painlevé property is possibly satisfied only if $A = 0$ and $p = -1$, the resonances being at $k = -1$ (generic) and $k = 4$. Consequently equation (5.7) now becomes

$$u'' + B\phi^m u + u^3 = 0 \qquad m = (4\sigma - 2)/(\sigma + 1) \tag{5.14}$$

provided $A = 0$, i.e.

$$2(\sigma + 1)^2 = 9\sigma(1 - r) \tag{5.15}$$

with the dominance being shared by u'' and u^3.

For $u(\phi)$ to be expanded in integral powers of ϕ the number m in equation (5.14) must be an integer. Note that a fractional power expansion [2] is not admissible since for u^3 the power is $\phi^{1/2}$ [2] which causes problems as it makes u complex and we are interested in real solutions. We thus have the following cases for which equation (5.7) possibly possesses the Painlevé property:

$$\begin{array}{ll} m = -1 & \sigma = 1/5 \\ m = 0 & \sigma = 1/2 \\ m = 1 & \sigma = 1 \\ m = 2 & \sigma = 2 \\ m = 3 & \sigma = 5 \end{array} \tag{5.16}$$

since we have $\sigma > 0$.

Now the full Painlevé analysis requires an expansion about an arbitrary point ϕ_0 in the complex ϕ-plane, i.e. we put

$$u = \sum_{n=0}^{\infty} a_n(\phi - \phi_0)^{n-1} \tag{5.17}$$

in

$$u''(\phi) + g(\phi)u(\phi) + u^3(\phi) = 0 \tag{5.18}$$

since at the point ϕ_0 we have a pole [2], and expand $g(\phi)$ as a Taylor series

$$g(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(\phi_0)(\phi - \phi_0)^n. \tag{5.19}$$

One finds [2] that $g''(\phi_0) = 0, \forall\phi_0$, for consistency. Hence $g(\phi) = \alpha + \beta\phi$, α and β being constants. On applying the aforementioned result to equation (5.14), where $g(\phi) = B\phi^m$, we deduce that for $B \neq 0$ equation (5.14) possesses the Painlevé property if $m = 0, \sigma = 1/2$ and $m = 1, \sigma = 1$ according to equation (5.16). Due to the validity of equation (5.15) we get

$$\begin{aligned} &\sigma = 1/2 \qquad b = 1 \qquad r = 0 \qquad (m = 0)\\ &\sigma = 1 \qquad b = 2 \qquad r = 1/9 \qquad (m = 1) \end{aligned} \tag{5.20}$$

for equation (5.14) to have the Painlevé property. For the (σ, b, r) values in (5.20) clearly the Lorenz equations (1.1)–(1.3) possess the Painlevé property, a result in accordance with previous analyses [25].

For $m = 0$ equation (5.14) becomes

$$u'' + Bu + u^3 = 0 \tag{5.21}$$

which is of the form of equation (3.13) and, thus, integrable through elliptic functions. We observe that the general solution of (5.21) and equations (2.8), (5.2) and (5.3) complemented by (2.3), (2.4) and (2.6) yield the general exact solution to the Lorenz equations (1.1)–(1.3) for $\sigma = 1/2$, $b = 1$, $r = 0$. This is the solution we already know from section 3 and have rederived here in an apparently simple manner. However, the approach of section 2 and the subsequent application of the Lie theory [5, 9] in section 3 yield besides the aforementioned general solution, in section 4 further possible integrable cases not following from a Painlevé analysis and also furnishes us with an analytical procedure for all r, b and σ with $b = 2\sigma$.

The case $m = 1$ in equation (5.14) is treated by letting

$$u(\phi) = \sum_{n=0}^{\infty} a_{n-1}\phi^{n-1}. \tag{5.22}$$

We obtain $a_{-1}^2 = -2$ (cf (5.13) and (5.15)), $a_0 = 0$, $a_1 = 0$, $a_2 = Ba_{-1}/4$ and a_3 is arbitrary as expected. The solution (5.22) corresponds to $\sigma = 1, b = 2, r = 1/9$ and is of no interest since $u(\phi)$ becomes complex. This again contrasts the procedure of sections 2–4, where for $\sigma = 1, b = 2, r = l/9$ the general exact solution to the Lorenz system can in principle be obtained in real form.

The case $B = 0$ corresponds to $C = 0$ (equation (5.6)) and

$$u'' + u^3 = 0. \tag{5.23}$$

Clearly the case $A = 0$, $B = 0$, owing to equations (2.8), (5.2), (5.3), (5.5), (5.6) and (5.50), is just the particular exact solution to equations (1.1)–(1.3) which was discussed in section 3.

It is now evident from the above that the Painlevé analysis of equation (5.7) produces either known results or complex valued $u(\phi)$. In the following we seek non-Painlevé particular solutions of equation (5.7). These solutions are represented in the form of power series. In the present paper we restrict ourselves to expansions motivated by the Painlevé findings. Hence, we consider equation (5.7) with $AB \neq 0$ for

$$\begin{aligned} &\sigma = 1 \qquad b = 2 \qquad r \neq 1/9 \\ &A = 9(-r + 1/9)/4 \qquad B = -4kC/3 \end{aligned} \tag{5.24}$$

we make the ansatz

$$u(\phi) = \sum_{n=0}^{\infty} a_n \phi^{3n-1}. \tag{5.25}$$

The expansion (5.25) originates from the Painlevé series (5.22) valid for $\sigma = 1, b = 2, r = 1/9$, where for $a_3 = 0$ the powers increase by multiples of three. Upon insertion of equation (5.25) into (5.7) we obtain

$$\begin{aligned} &a_0^2 = -(A + 2) \\ &a_1 = Ba_0/[2(A + 2)] \\ &a_2 = B^2 a_0/[2^3(7 - A)(2 + A)] \\ &a_3 = B^2 a_0(11 + A)/[2^4(A + 2)^2(25 - A)(7 - A)], \text{etc.} \end{aligned} \tag{5.26}$$

We get, therefore, a series proceeding in powers of ϕ^3 as

$$u(\phi) = a_0\phi^{-1}\left\{1 + \frac{B\phi^3}{2(A+2)} + \frac{B^2\phi^6}{2^2(A+2)(7-A)} + \frac{B^3\phi^9(11+A)}{2^4(A+2)^2(7-A)(25-A)} + \cdots\right\}. \tag{5.27}$$

By virtue of equations (2.8), (5.2), (5.3) and (5.27)

$$\begin{aligned} &x(t) = -a_0(2^{3/2})3^{-1}k^{1/2}\left\{1 + \frac{B\phi^3}{2(A+2)} + \cdots\right\} \\ &\phi(t) = -3\exp(-2t/3)(16k)^{-1/3}. \end{aligned} \tag{5.28}$$

Since we require $x(t)$ given by equation (5.28) to be real, it follows from equations (5.24) and (5.26) that $A < -2$, i.e. $r > 1$.

For $t \to \infty$ equation (5.28) shows that $x \to -2^{3/2}a_0k^{1/2}/3\text{a}$. Since the nonzero equilibrium points of the system (1.1)–(1.3) for ($\sigma = 1, b = 2, r > 1$) are $(\pm\sqrt{2(r-1)}, \pm\sqrt{2(r-1)}, r-1)$, it is clear that due to equations (5.24) and (5.26) we must choose $k = 1$ in the expression $-2^{3/2}a_0k^{1/2}/3$. Thus the constant k is fixed in equation (5.28). Regarding convergence of the series in (5.28) we observe that we have an expansion in $B\phi^3 = 9C[\exp(-2t)]/4$ ($k = 1$). Therefore the above series converges for all C if t is sufficiently large or, equivalently, for all t if C is small enough.

We conclude that equations (5.28), (2.3), (2.4) and (2.6) constitute two particular non-Painlevé exact solutions to the system (1.1)–(1.3) depending on one constant of integration C and valid for $k = 1$ in equations (5.2) and (5.3) and $\sigma = 1, b = 2, r > 1$. The above solutions are generated by the two different signs of a_0 in equation (5.26) and approach the equilibrium points $(\sqrt{2(r-1)}, \sqrt{2(r-1)}, r-1)$ for $a_0 = -\frac{3}{2}\sqrt{r-1}$ and $(-\sqrt{2(r-1)}, -\sqrt{2(r-1)}, r-1)$ for $a_0 = \frac{3}{2}\sqrt{r-1}$ respectively.

Next we consider

$$\begin{aligned} &\sigma = \tfrac{1}{2} \qquad b = 1 \qquad r \neq 0 \\ &A = -2r \qquad B = (2k)^{2/3}C/2. \end{aligned} \tag{5.29}$$

The appropriate expansion is now

$$u(\phi) = \sum_{n=0}^{\infty} a_n \phi^{2n-1}. \tag{5.30}$$

Proceeding as in equation (5.24) we deduce two further particular non-Painlevé exact solutions of the system (1.1)–(1.3) depending on one integration constant C and holding for $k = 1$ in equations (5.2) and (5.3) and $\sigma = \frac{1}{2}$, $b = 1$, $r > 1$. These solutions approach the equilibrium points $(\sqrt{(r-1}, \sqrt{r-1}, r-1)$ for $a_0 = -\sqrt{2(r-1)}$ and $(-\sqrt{r-1}, -\sqrt{r-1}, r-1)$ for $a_0 = \sqrt{2(r-1)}$ respectively and are given by equations (2.3), (2.4) and (2.6) with

$$x(t) = -a_0 \left\{ 1 + \frac{B\phi^2}{2(A+3)} - \frac{B^2\phi^4}{2^3(A+3)^2} - \frac{B^3\phi^6}{2^4(A+3)^3(7-A)} + \cdots \right\}. \tag{5.31}$$

In equation (5.31) the series proceeds in powers of $B\phi^2 = 2C\exp(-t)$ and is convergent subject to the same conditions as in the case of the expansion (5.28). Note that the value $A = -3$ must be excluded as it leads in the course of the calculation of a_n in equation (5.30) to $a_0 = 0$ and thus to the trivial $u(\phi) = 0$.

For the sake of completeness we have also investigated the (m, σ) cases in equation (5.16) for which (5.7) does not possess the Painlevé property. To avoid repetitious material we merely state our results. We obtain precisely, as in the cases (5.24) and (5.29), $x(t)$ in the form of series and, consequently, particular exact solutions of the Lorenz equations approaching $(\pm\sqrt{2\sigma(r-1)}, \pm\sqrt{2\sigma(r-1)}, r-1)$.

We now finish our investigation of equation (5.7) by submitting it to a Lie analysis. Writing for brevity $m = (4\sigma - 2)/(\sigma + 1)$ (equation (5.14)) we find that (5.7) becomes

$$u''(\phi) + \left(\frac{A}{\phi^2} + B\phi^m\right) u(\phi) + u^3(\phi) = 0. \tag{5.32}$$

To obtain more general results we seek the functions $g(\phi)$ for which the differential equation

$$u''(\phi) + g(\phi)u(\phi) + u^3(\phi) = 0 \tag{5.33}$$

possesses the symmetry

$$G = \xi(\phi, u)\partial_\phi + \eta(\phi, u)\partial_u. \tag{5.34}$$

By following the method applied in section 3 we obtain that $\xi(\phi, u)$ and $\eta(\phi, u)$satisfy

$$\frac{\partial^2\xi}{\partial u^2} = 0 \tag{5.35}$$

$$\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi}{\partial\phi\partial u} = 0 \tag{5.36}$$

$$2\frac{\partial^2\eta}{\partial\phi\partial u} + 3[g(\phi)u(\phi) + u^3(\phi)]\frac{\partial\xi}{\partial u} - \frac{\partial^2\xi}{\partial\phi^2} = 0 \tag{5.37}$$

$$\frac{\partial^2\eta}{\partial\phi^2} - [g(\phi)u(\phi) + u^3(\phi)]\frac{\partial\eta}{\partial u} + 2[g(\phi)u(\phi) + u^3(\phi)]\frac{\partial\xi}{\partial\phi} + \xi g'(\phi)u'(\phi) + g(\phi)\eta + 3\eta u^2(\phi) = 0. \tag{5.38}$$

From equation (5.35)

$$\xi(\phi, u) = a(\phi) + b(\phi)u(\phi). \tag{5.39}$$

From equation (5.36)

$$\eta(\phi, u) = b'(\phi)u^2(\phi) + c(\phi)u(\phi) + d(\phi). \tag{5.40}$$

Owing to equations (5.37), (5.39) and (5.40)

$$b(\phi) = 0 \qquad a''(\phi) = 2c'(\phi) \tag{5.41}$$

and thus, due to equations (5.41), we obtain from (5.38)

$$d(\phi) = 0 \qquad c(\phi) + a'(\phi) = 0 \qquad c''(\phi) + 2a'(\phi)g(\phi) + a(\phi)g'(\phi) = 0. \tag{5.42}$$

Equations (5.42) yield

$$c(\phi) = C_0 \qquad a(\phi) = A_0 - C_0\phi$$
$$g(\phi) = \frac{K_0}{(A_0 - C_0\phi)^2} \tag{5.43}$$

C_0, A_0, K_0 being constants. The general result (5.43) implies that equation (5.32) has the symmetry (5.34) provided $B = 0$, where also $A = K_0$, $A_0 = 0$ and $C_0 = -1$. The condition $B = 0$ shows that in equation (5.6) we must take

$$C = 0. \tag{5.44}$$

Consequently equation (5.32) becomes

$$u''(\phi) + \left(\frac{A}{\phi^2}\right) u(\phi) + u^3(\phi) = 0 \tag{5.45}$$

and possesses the symmetry

$$G = \phi\partial_\phi - u\partial_u. \tag{5.46}$$

By means of the transformation (cf section 3)

$$Y = u\phi \qquad X = \log(-\phi) \tag{5.47}$$

equation (5.45) is written as

$$Y'' - 3Y' + (A+2)Y + Y^3 = 0 \tag{5.48}$$

where the prime denotes differentiation with respect to X. Excluding the case $A = 0$ in equation (5.45), which leads to the particular exact solution of the Lorenz equations discussed in section 3, we observe that equation (5.48) has the singular solution

$$Y(X) = \pm(-A-2)^{1/2} \tag{5.49}$$

and this is real provided $A < -2$, which yields by equation (5.5), $r > 1$. From equations (2.8), (5.2), (5.3), (5.47) and (5.49)

$$x(t) = \pm\tfrac{1}{3}[-A-2]^{1/2}[(\sigma+1)(2k)^{1/2}]. \tag{5.50}$$

By choosing in equation (5.50) $k = 1$ it is clear that we obtain the nonzero constant solution of the Lorenz equations where the initial conditions are the equilibrium points $(\pm\sqrt{2\sigma(r-1)}, \pm\sqrt{2\sigma(r-1)}, r-1)$, $r > 1$.

Having dispensed with the singular solution of equation (5.48) we employ the substitution

$$w(Y) = \frac{\mathrm{d}Y(X)}{\mathrm{d}X} \tag{5.51}$$

to reduce equation (5.48) to an Abel's equation of the second kind, namely

$$w\dot{w} - 3w + (A+2)Y + Y^3 = 0 \tag{5.52}$$

the dot standing for $\mathrm{d}/\mathrm{d}Y$. We have treated equation (5.52) in detail previously [3, 4]. On the basis of equations (2.3), (2.4), (2.6), (2.8), (5.2) and (5.3) and following the analysis of [3, 4] we may prove the existence of two particular exact solutions $(x(t), y(t), z(t))$ of the Lorenz equations, monotonic decreasing, valid for $r > 1$ and approaching $(\pm\sqrt{2\sigma(r-1)}, \pm\sqrt{2\sigma(r-1)}, r-1)$. These solutions depend on one constant of integration and have essentially the same structure as the monotonic decreasing solutions found earlier [4]. It is noteworthy that on endowing equation (2.7) with condition (5.44) and subsequently investigating it in the context of the theory of nonlinear differential equations [3, 4] we obtain practically the same results as are found by transforming equation (2.7) to (5.7) and applying the Lie theory [5, 9].

6. Conclusions

The objective of this paper has been to seek exact solutions of the Lorenz equations. By transforming the Lorenz system for $b = 2\sigma$ to a generalized Emden–Fowler equation and subsequently applying the Lie theory of extended groups we found in closed form the general exact solution of the Lorenz equations for $\sigma = \frac{1}{2}$, $bb = 1$, $r = 0$. Alternatively by reducing the Lorenz system for $b = 2\sigma$ to a time-dependent oscillator with constant coefficient anharmonic term and by employing the Painlevé analysis we have rederived the general exact solution mentioned above. The appropriate extension of the Painlevé method results provides in the form of power series further particular exact solutions not possessing the Painlevé property. As a final remark we stress that, although the Lie analysis and the Painlevé approach seem to complement each other, the Lie theory appears to be instrumental in an attempt for the construction of the general exact solution of the Lorenz equations for all $b = 2\sigma$.

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